$$\rho\,(u_t + uu_x) = \sigma_x,\ \rho_t + (\rho u)_x = 0$$

hold (in $L_2(Q_\delta)$). Since the function $v(x,t)$ has for any t for almost all x a finite derivative v_x, then from the equation $\sigma = \mu u_x - p$ it follows that the finite derivate u_{xx} exists almost everywhere. The theorem is proved.

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LOCALIZATION OF GAS-DYNAMIC PROCESSES AND STRUCTURE WHEN THE MATERIAL IS COMPRESSED ADIABATICALLY, IN THE PEAKING MODE*

A.P. MIKHAILOV and V.V. STEPANOVA

Adiabatic compression of gas by a piston, the pressure on which increases in the peaking mode, is studied. The entropy is distributed over the mass. A class of selfsimilar solutions (the *LS* mode) is constructed and its properties are studied. It is shown that the effective dimensions of the compression wave decrease with time and all gas-dynamic perturbations are localized within a finite mass of the gas. The solutions obtained are characterized by the presence of a structure (inhomogeneities) in the density and temperature. The compression occurs without the formation of shock waves.

The peaking mode, i.e. the precesses in which any quantities may become infinite in a finite period of time, have a number of unusual properties. Thus the development of the peaking modes in continua is accompanied by localization ("inertia") of the diffusion processes and the formation of non-stationary dissipative structures /1-3/.

Another example is offered by an isentropic (optimal) compression of a finite mass of gas to superhigh densities /2,4-7/**. Such a process takes place when the pressure acting on the compressing piston increases as follows (the *S* mode):

$$P(0,\ t) = P_0\,(t_f - t)^n,\quad n = -2\gamma\,(N+1)/(2 + (N+1)(\gamma - 1));$$
$$t_0 \leqslant t \leqslant t_f$$

where $N = 0, 1, 2$ is a geometrical index, γ is the adiabatic index and t_f denotes the instant of peaking.

The problem of the adiabatic compression of a cold gas initially at rest, by a piston acted upon by a pressure which varies with time according to a more general law, with peaking at any $n < 0$, is considered below for the case when $N = 0$.

Another generalization consists of the fact that the entropy of the gas depends on the Lagrangian mass coordinate $x \geqslant 0$ is such a manner, that $P(x,\ t) = a_0 x^\delta \rho^\gamma$ for all $t_0 \leqslant t < t_f$. Such a distribution of entropy in the medium arises e.g. behind the shock wave front moving through the gas, with velocity varying with time according to a power law.

Selfmodelling solutions are constructed for $n > -2\gamma/(\gamma + 1)$ (the *LS* mode) corresponding to

** See also: Kazhdan Ya.M. On the problem of adiabatic compression of gas by a spherical piston. Preprint In-ta prikl. matem. Akad. Nauk SSSR, Moscow, No.89, 1975.

*Prikl.Matem.Mekhan., 48,6,921-928,1984

a pressure growth "slower" than that in the S mode $(n = -2\gamma/(\gamma+1))$. Numerical computations using the "FLOR" /8/ program were used to demonstrate the stability of the solutions obtained.

1. Formulation of the selfsimilar problem.

Consider one-dimensional (plane) non-steady adiabatic gas flows described by the following set of equations:

$$\frac{\partial}{\partial t}\left(\frac{1}{\rho}\right) = \frac{\partial U}{\partial x}, \quad \frac{\partial U}{\partial t} = -\frac{\partial P}{\partial x}, \quad P = a_0 x^\delta \rho^\gamma, \quad t_0 \leqslant t < t_f \tag{1.1}$$

where $x \geqslant 0$ is the Lagrangian mass coordinate and $U(x, t), \rho(x, t), P(x, t)$ are the velocity, density and pressure of the gas respectively.

The gas is set in motion by means of a piston situated at the point $x = 0$, and the pressure on it varies within the peaking mode

$$P(0, t) = P_0 (t_f - t)^n, \ n = \text{const} < 0, \ t_0 \leqslant t < t_f \tag{1.2}$$

The passage to a problem with velocity boundary conditions is discussed at the end of Sect. 4.

The gas is at rest at the initial instant

$$U(x, t_0) = 0, \ 0 \leqslant x \leqslant \infty \tag{1.3}$$

Dimensional analysis implies that in order to formulate the selfsimilar problem we must put $t_0 = -\infty$ and eliminate the pressure (or density) parameter from the initial data

$$P(x, t) = a_0 x^\delta \rho^\gamma (x, t) \to 0, \ t \to -\infty, \ 0 \leqslant x \leqslant \infty \tag{1.4}$$

The selfsimilar formulation corresponds to the situation where the piston is present at the initial instant at the point at infinity ($r(0, t)$ is the radius of the piston)

$$r(0, t) \to -\infty, \ t \to -\infty$$

In accordance with dimensional analysis the solution of problem (1.1)-(1.4) can be written in the form ($t_f = 0$ without loss of generality)

$$\begin{aligned} &P(x, t) = P_0 (-t)^n \pi(\xi), \rho(x, t) = \rho_0 (-t)^k g(\xi), \ U(x, t) = \\ &\quad U_0 (-t))^l v(\xi) \\ &k = ((2-\delta) n - 2\delta)/(2\gamma+\delta), \ l = ((\gamma - 1 + \delta) n + \delta)/(2\gamma + \delta) \\ &U_0 = (P_0^{\gamma-1+\delta} a_0)^{1/(2\gamma+\delta)}, \ \rho_0 = (P_0^{2-\delta} a_0^{-2})^{1/(2\gamma+\delta)} \end{aligned} \tag{1.5}$$

and the selfsimilar coordinate is

$$\xi = x/(x_0 (-t)^m), \quad x_0 = (P_0^{\gamma+1} a_0^{-1})^{1/(2\gamma+\delta)}, \quad m = ((\gamma+1) n + 2\gamma)/(2\gamma+\delta) \tag{1.6}$$

For the selfsimilar functions the problem becomes

$$\begin{aligned} &m\xi g' + g^2 v' = kg, \quad \gamma \xi^\delta g^{\gamma-1} g' + m\xi v' = lv - \delta g^\gamma \xi^{\delta-1}, \quad \pi = \xi^\delta g^\gamma \\ &\pi(0) = 1, \ \pi(\xi_f) = v(\xi_f) = 0 \ (0 < \xi_f \leqslant \infty) \end{aligned} \tag{1.7}$$

Here ξ_f is the coordinate of the wave front, i.e. of the point separating the region in motion from the unperturbed gas.

We will seek a continuous, non-negative solution of (1.7).

The functions $\pi(\xi), v(\xi)$ are bounded and non-negative for all $0 \leqslant \xi \leqslant \infty$, and the function $g(\xi)$ is either zero $(\delta < 0)$ or infinite $(\delta > 0)$ at the point $\xi = 0$.

We note that the problem formulated above can also be considered in the case when $t_0 > -\infty$, provided that $P(x, t_0), U(x, t_0)$ are given in the form (1.5) where the functions $\pi(\xi), v(\xi)$ satisfy problem (1.7).

2. The conditions for a solution to exist.

Analysis of problem (1.7) yields the following asymptotic forms: near the piston $(\xi = 0)$ we have

$$\begin{aligned} &g(\xi) = \xi^{-\delta/\gamma} (1 + C_{g2}\xi + \ldots) \\ &v(\xi) = v(0) + \frac{n}{\gamma+\delta} \xi^{(\gamma+\delta)/\gamma} + \ldots \\ &\pi(\xi) = 1 + C_{\pi 2}\xi + C_{\pi 3}\xi^{(2\gamma+\delta)/\gamma} + \ldots \\ &v(0) = C_{\pi 2}/l > 0, \ C_{g2} = C_{\pi 2}/\gamma \end{aligned} \tag{2.1}$$

and near the front ($\xi_{fr} = \infty$ the assumption that $\xi_{fr} < \infty$ leads to a contradiction)

$$\begin{aligned} &g(\xi) = \xi^{k/m} (C_{g1} + C_{g2}\xi^{-1/m} + \ldots) \\ &v(\xi) = \xi^{l/m} (C_{v1} + C_{v2}\xi^{-1/m} + \ldots) \end{aligned} \tag{2.2}$$

$$\pi(\xi) = \xi^{n/m}(C_{\pi 1} + C_{\pi 2}\xi^{-1/m} + \ldots)$$
$$C_{v1} = 2mC_{g3}/(lC_{g1}^2),\ C_{v2} = nC_{g1}^{\gamma}/m$$

The constants accompanying the first terms of the expansion (2.2) are positive, and those of the second terms are negative. Their values depend on n, δ, γ and are found from the numerical solution of problem (1.7).

The necessary conditions for solutions to exist follow from the boundary conditions of (1.7) and the asymptotic forms (2.1), (2.2)

$$\delta > -\gamma,\ k < 0,\ l < 0,\ m > 0 \tag{2.3}$$

The solutions of the problem are segregated according to the sign of the parameter m. The case of $m = 0$ (separation of variables) corresponds to the S mode, and the case of $m > 0$ to the LS mode. When $m < 0$ (the HS mode in accordance with the terminology of /1-3/). There is no solution.

Numerical experiment shows that the class of boundary HS modes is characterized by the fact that their interaction with the medium generates a compression wave (which must contain a discontinuity). As $t \to t_f$, the wave embraces all the material and there is no localization.

3. Construction of the solution.

By making the change of variables

$$\eta = \ln \xi,\ \pi(\xi) = \xi^{(2\gamma+\delta)/(\gamma+1)}P(\eta), \tag{3.1}$$
$$g(\xi) = \xi^{(2-\delta)/(\gamma+1)}G(\eta),\ v(\xi) = \xi^{(\gamma-1+\delta)/(\gamma+1)}V(\eta)$$

we reduce the system for the selfsimilar functions g, v, π to the form, autonomous in ξ

$$dV/d\eta = \Delta_V/\Delta,\ dG/d\eta = \Delta_G/\Delta \tag{3.2}$$
$$\Delta_V = \frac{\gamma-1+\delta}{\gamma+1}\gamma VG^{\gamma+1} - \frac{\gamma-1}{\gamma+1}mV - nG^{\gamma}$$
$$\Delta_G = \frac{2\gamma+\delta}{\gamma+1}G^{\gamma+2} - \frac{2m}{\gamma+1}G - lVG^2,\ \Delta = m^2 - \gamma G^{\gamma+1}$$

Problem (1.7) reduces to an ordinary, first-order differential equation

$$dV/dG = \Delta_V/\Delta_G \tag{3.3}$$

with boundary conditions $V(G=0) = 0$; $V(G=\infty) = \infty$, $\delta > 1-\gamma$; $V(G=\infty) = 0$, $-\gamma < \delta < 1-\gamma$, obtained using (2.2), (2.3).

Every value of the variable η (or ξ) must be in 1:1 correspondence with a unique value of the solution (the functions V and G); therefore the functions $\eta(V)$ and $\eta(G)$ must be monotonic in the solution

$$d\eta/dG \neq 0,\ d\eta/dV \neq 0,\ G \geqslant 0,\ V \geqslant 0$$

The required solution connecting the points $G = 0$ and $G = \infty$ in the plane VG (see (3.3)), must pass through the line $G_* = (m^2/\gamma)^{1/(\gamma+1)}$ on which the product of $d\eta/dG$ and $d\eta/dV$ changes sign, apart from the points where $\Delta_V = 0$, $\Delta_G = 0$. Consequently, it is necessary, for uniqueness, that the solution should pass through the singularity F_1 of (3.3), defined by the conditions

$$\Delta_G = \Delta_V = \Delta = 0 \tag{3.4}$$
$$F_1 = \{G_* = (m^2/\gamma)^{1/(\gamma+1)},\ V_* = mn/(\gamma l G_*)\}$$

The points F_1 is a saddle point with critical directions

$$U_{1,2} = (dV/dG)_{1,2} = [m(n(\gamma+1) + 4\gamma + \delta)/(2\gamma+\delta) \pm [m^2(n(\gamma+1) + 4\gamma+\delta)^2/(2\gamma+\delta)^2 - 4m^2n(n(\gamma-1+\delta) + 2\gamma + \delta - 2)/\gamma]^{1/2}]/(2lG_*^2) \tag{3.5}$$

The points lying on the wave front and the piston, represent the singularities of (3.3).

The wave front appears at the point A $(G = 0, V = 0)$, nodal for any value of the parameters $n, \delta, \gamma > 1$. The integral curves near the front have the form

$$V = C_1(n, \delta, \gamma)G^{(\gamma-1)/2} + C_2(n, \delta, \gamma)G^{\gamma} + \ldots$$

When $-\gamma < \delta < 1-\gamma$, the point B $(V = 0, G = \infty)$ corresponds to the piston. At the given values of δ the point B is a node. The solution enters the singularity along the $0G$ axis

$$V = C_1(n, \delta, \gamma)G^{\lambda} + n/(\gamma+\delta)G^{-1} + \ldots,\ \lambda = \gamma(\gamma - 1 + \delta)/(2\gamma+\delta)$$

When $\delta > 1-\gamma$, the piston is at the point C $(V = \infty, G = \infty)$ representing a complex singularity. Near the point the integral curves have the form

$$V = 1/lG^{\gamma} + O(G^{\mu}),\ \mu = \gamma(\delta - 2)/(\gamma - 1 + \delta); \tag{3.6}$$
$$V = C(n, \delta, \gamma) G^{\lambda}$$

where $C(n, \delta, \gamma)$ are found from the numerical solution of the problem. The second curve of (3.6) satisfies the boundary condition.

We construct the field of integral curves by inspecting the regions of monotony and behaviour of the solution near the singularities. The solution of the problem must connect the points corresponding to the front and the piston, and pass through the point F_1 (see (3.4)). Then the conditions are satisfied by one of the separatrices of the point F_1, and the separatrix is the required solution.

Figure 1 shows the field of integral curves for the case $-\gamma < \delta < 1 - \gamma$, $-2\gamma/(\gamma + 1) < n < 2\delta/(2 - \delta)$. The solid line is the solution, the dashed lines represent the isoclines of the zero (V_0) and infinity (V_∞), and the dot-dash line is the line $G_* = (m^2/\gamma)^{1/(\gamma+1)}$.

Construction of the regions of monotony of the functions g, v, π, T in the VG plane, leads to the following conclusions.

The pressure and ensity are monotonically decreasing functions of the selfsimilar variable ξ, and hence of the mass coordinate x. The density (temperature) is a non-monotonic function of x when $\delta < 0$ $(\delta > 0)$. The density and temperature have at most a single maximum, which follows from the third equation of (1.1).

We thus have

Theorem. A solution of problem (1.7) exists and is unique when $\delta > -\gamma$ and $-2\gamma/(\gamma + 1) < n < -\delta/(\gamma - 1 + \delta)$ $(\delta > 1 - \gamma)$, $-2\gamma/(\gamma + 1) < n < 2\delta/(2 - \delta)$ $(-\gamma < \delta < 1 - \gamma)$. The boundary conditions in (1.7) hold when $\xi_{fr} = \infty$. The functions $\pi(\xi)$, $v(\xi)$ are monotonic function $g(\xi)$ $(T(\xi))$ is monotonic when $\delta > 0$ $(\delta < 0)$ and has a unique maximum when $\delta < 0$ $(\delta > 0)$. When $n < -2\gamma/(\gamma + 1)$ and $n > -\delta/(\gamma - 1 + \delta)$ $(\delta > 1 - \gamma)$, $n > 2\delta/(2 - \delta)$ $(-\gamma < \delta < 1 - \gamma)$, there is no solution.

Notes. 1°. Depending on the magnitude of δ, a solution does not exist for all values of n corresponding to the *LS* mode.

2°. The piston radius $r(0, t)$ varies from $-\infty$ (at the "initial" instant $t = -\infty$), to zero (at the instant of focusing).

Figure 2 shows graphs of the functions $g(\xi)$, $\pi(\xi)$ for $n = -1.22$, $\delta = -1.2$, $\gamma = 5/3$ obtained by a numerical solution of the problem (1.7), when $t_1 = -0.40$ (the light circles), $t_2 = -1.02 \cdot 10^{-2}$ (the dark circles) and $t_4 = -9.05 \cdot 10^{-4}$ (the crosses).

4. Physical properties of the solution.

Contraction of the effective dimensions. The effective dimensions of the wave (the half-width and other points with a fixed selfsimilar coordinate, e.g. the position of the maxima), contract with time, approaching the piston as given by

$$x_{ef} \sim \xi_{ef}(-t)^m \to 0,\ t \to -0\ (P(x_{ef}(t), t) = {}^1/_2 P(0, t)) \tag{4.1}$$

The compression wave front is situated at the point at infinity, but this does not contradict the finite velocity of perturbations, since infinite time elapses between the start of the process $t_0 = -\infty$ and the instant $t > t_0$, otherwise the perturbed region would contract with time.

The energy imparted to the gas by the piston enters a continuously contracting region.

Near the initial instant of time, in the neighbourhood of the piston (front) we have finite (infinite) energy when $n > -2(\gamma + \delta)/(3\gamma - 1 + \delta)$ and infinite (finite) energy when $-2\gamma/(\gamma + 1) < n < -2(\gamma + \delta)/(3\gamma - 1 + \delta)$.

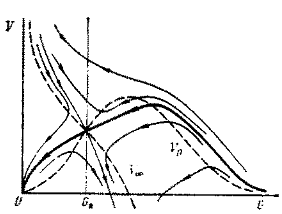

Fig. 1

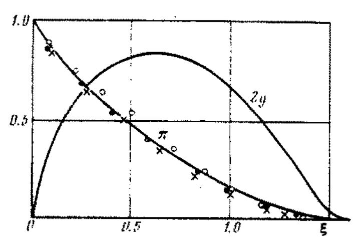

Fig.2

On approaching the instant of focusing, the energy enters the contracting gaseous region near the piston. When $n > -2(\gamma + \delta)/(3\gamma - 1 + \delta)$, the energy is finite, and it becomes infinite when $-2\gamma/(\gamma+1) < n < -2(\gamma+\delta)/(3\gamma-1+\delta)$. This follows from relations (2.1) (2.2) and the energy equation

$$\frac{\partial}{\partial t}(1/(\gamma+1)P+\rho U^2/2) = -\frac{\partial}{\partial r}(PU)$$

The wave contains an "acoustic" point at which the Lagrangian speed of sound is equal to the rate of propagation of the fixed, selfsimilar state (in the *VG* plane we have the corresponding singularity F_1 on the "acoustic" line $G_* = (m^2/\gamma)^{1/(\gamma+1)}$). The flow is "supersonic" between the "acoustic" point and the front, and "subsonic" between the front and the singularity. Unlike the regions without peaking, the acoustic singularity is traversed in a continuous manner.

Limiting curves and localization. Although the pressure and velocity of the piston both increase in the peaking mode, localization of gas-dynamic processes nevertheless occurs.

Indeed, for every fixed $0 < x^* < \infty$ the quantity $\xi^* = x^*/(x_0(-t)^m) \to \infty$ as $t \to -0$. Then, using relation (2.2) we obtain the following asymptotic expressions as $t \to -0$:

$$\begin{aligned} P(x,t) &= C_{P1}x^{n/m} + C_{P2}x^{(n-1)/m}(-t) + \ldots, \quad C_{P2} < 0 \\ U(x,t) &= C_{u1}x^{l/m} + C_{u2}x^{(l-1)/m}(-t) + \ldots, \quad C_{u2} < 0 \\ R(x,t) &= C_{R1}x^{(l+1)/m} + C_{R2}x^{l/m}(-t) + \ldots, \quad C_{R2} < 0 \end{aligned} \tag{4.2}$$

Here $R(x,t)$ is the distance between the piston and the point with mass coordinate x.

System (1.1) and the monotonic form of the function π, v imply that for fixed x^* all quantities ρ, U, P, T are monotonic with respect to time.

Thus in the *LS* mode every function ρ, U, P, T has its limit curve, in other words every function has a constant upper limit. As $t \to -0$, the solution approaches the limit curve from below as given by (4.2). In the *LS* mode we have localization of the gas-dynamic processes, and any fixed physical state does not penetrate past a certain finite mass of gas.

Gas-dynamic structures. The distribution of entropy over the mass of material stipulates the presence of localized maxima of the density and temperature (gas-dynamic structures) in the compression wave.

The maxima of the structures are compressed to the piston with time as given by a law similar to (4.1). Density structures exist when $\delta < 0$, and temperature structures when $\delta > 0$. When $\delta = 0$ all the functions are monotonic and there are not structures.

The degree of compression (heating) of the portion of the medium is determined by its entropy, and the pressure within it. Therefore, as the solutions constructed show, high densities (temperatures) can be reached in regions with lower pressure when the pressure profile in the compression wave is monotonic. The internal cause of the structure is the localization ("time lag") of the gas-dynamic processes, which occurs when the peaking modes develop in the medium /9, 10/*, and this conbines them with the non-stationary dissipative structures /3/.

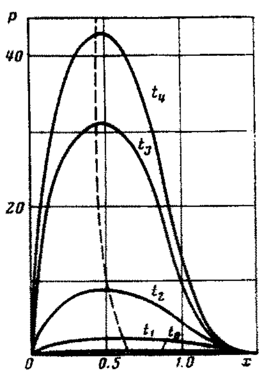

Fig.3

Example of the numerical solution. Numerical computations for system (1.1) (Fig. 3) illustrate the stability of the solutions constructed. The profiles of the quantities near the selfsimilar solution of the problem with $n=-1.22$, $\gamma=5/3$, $\delta=-1.2$ (see Fig. 2) at the instant of time $t_0=-1$ $(a_0=P_0=1)$ are used as the initial data.

Figure 3 shows the distribution of gas density over the mass at various instants of time $t_0=-1$, $t_1=-0.40$, $t_2=-1.02\cdot10^{-2}$, $t_3=-1.46\cdot10^{-3}$, $t_4=-9.05\cdot10^{-4}$, with the dashed line showing the position of the maxima. Figure 2 shows the profiles of the selfsimilar function π (of pressure) obtained when processing the numerical solution of the problem (1.1)-(1.4) in accordance with relations (1.5), at various instants of time. The results obtained show that the selfsimilar pressure profile is reproduced when the pressure at the piston is increased by a factor of 10^4.

The problem with a velocity at the piston. If the variation in velocity is specified at the piston in the mode with peaking

$$U = U_0(-t)^{n_v}, \quad n_v < 0, \quad U_0 > 0$$

* see also: Demidov M.A. and Mikhailov A.P: Localization and structures during the adiabatic compression of a finite mass of gas in the peaking mode, Preprint In-ta prikl. matem. Akad. Nauk SSSR, No. 8, 1983.
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the problem is equivalent to that discussed above. The transition is effected by simple recomputation using the formulas

$$m=\frac{(\gamma+1)n_v+\gamma+1}{\gamma-1+\delta},\quad n=\frac{(2\gamma+\delta)n_v-\delta}{\gamma-1+\delta},\quad k=\frac{(2-\delta)n_v-\delta}{\gamma-1+\delta}$$

The solution of the initial problem (with completely analogous properties) will exist for the following corresponding values of the parameters: δ: $\delta/(2-\delta)<n_v<(1-\gamma)/(1+\gamma)$, $-\gamma<\delta<1-\gamma$; $(1-\gamma)/(\gamma+1)<n_v<0$, $\delta>1-\gamma$.

"Inversion" of the solutions with time. Let the piston move out of the gas and $P(0,t)=P_0(-t)^n$, and let the time vary within the range from $t=t_0$ to $t=-\infty$, this being equivalent to the relation $P(0,t)=P_0t^n$, $0\leqslant t_0<\infty$.

If we take the solution of (1.7) at $t=t_0$ as the initial data, with the sign of the velocity changed, then the spatial profiles of the quantities in both problems will coincide. The solution represents a rarefaction wave with the half-width increasing according to the selfsimilar law. If $t_0=0$, then the limit curves (4.2) become the initial data.

Inversion of the solutions inthe case of the *S* mode was discussed in /11/.

The solutions obtained are characterized by the lack of discontinuities, contraction of effective dimensions of the compression wave and localization of the gas-dynamic perturbations consisting of the fact that a state with any quantity (pressure, velocity, density) fixed does penetrate outside a certain, finite mass of gas, even when the pressure at the piston increases without limit as $t\to t_f$. When $x>0$, any quantity is bounded from above by the corresponding limit curve.

The solutions contain the gas-dynamic structures (localized maxima) of the density $(\delta<0)$ of temperature $(\delta>0)$, and the maxima are attracted towards the piston as $t\to t_f$. When $\delta=0$ we have no isentropic case /9/ of the structure. The pressure and velocity are monotonic for any values of δ.

When $n<-2\gamma/(\gamma+1)$, the "faster" *HS* mode), the problem in question has no selfsimilar solution. The localization and structure were studied in the case of the *S* mode in /10/.

The properties of the solution shown here are essentially different form those in modes without peaking /12-15/ and indicate the possibility of realizing various, physically different methods of compressing the material and controlling the process.

The results of the present paper and of /9-10/* enable us to conclude that, irrespective of the different nature of the diffusion and the gas-dynamic processes, a general law exists governing peaking modes in continua.

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ON THE OPTIMAL CONTROL OF VISCOUS INCOMPRESSIBLE FLUID FLOW*

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The framework of the Navier-Stokes (N-S) equations is used to study flow past an arbitrary body on whose surface the tangential or normal velocity is under control. The necessary conditions are obtained for the minimum rate of energy dissipation. Exact analytical solutions of the corresponding problems are found for the case of flow past an ellipsoid in the Stokes approximation.

1. Let a body S by streamlined by a stationary flow of a viscous incompressible fluid. We shall consider the following variational problem: to find a suction (injection) velocity distribution over the body surface, for which the rate of energy dissipation D is minimal. We shall assume here that the total flow of fluid across the surface of S is zero.

Using dimensionless variables we write the equations of motion for the fluid, the boundary conditions and the minimizing functional in the form

$$\Delta \mathbf{V} - \nabla p - R(\mathbf{V}\cdot\nabla)\mathbf{V} = 0, \quad \nabla\cdot\mathbf{V} = 0, \quad \mathbf{V}|_S = Wn, \quad \mathbf{V}|_\infty = \mathbf{U} \tag{1.1}$$

$$D(W) = \int_\Omega \frac{1}{2} \sum_{i,j=1}^{3} \left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)^2 d\Omega \tag{1.2}$$

where Ω is the outside of the body S, $\mathbf{n} = (n_1, n_2, n_3)$ is the unit vector of the external normal, $\mathbf{U}$ is the stream velocity at infinity and R is the Reynolds number. The N-S equations are made dimensionless so as to ensure their simplest form in the limiting case of the Stokes flow as $R \to 0$.

2. To obtain the necessary condition for the minimum of the functional (1.2), we shall write the rate of suction (injection) W, the rate of flow $\mathbf{V}$ and the pressure p in the form

$$W = W_0 + \varepsilon W_1, \quad V = V_0 + \varepsilon V_1 + O(\varepsilon^2) \tag{2.1}$$
$$p = p_0 + \varepsilon p_1 + O(\varepsilon^2), \quad 0 < \varepsilon \ll 1$$

The functions W_0, $\mathbf{V}_0$ and p_0 satisfy the boundary condition (1.1), while W_1, $\mathbf{V}_1$ and p_1 satisfy the boundary value problem

$$\Delta\mathbf{V}_1 - \nabla p_1 - R[(\mathbf{V}_0\cdot\nabla)\mathbf{V}_1 + (\mathbf{V}_1\cdot\nabla)\mathbf{V}_0] = 0, \quad \nabla\cdot\mathbf{V}_1 = 0 \tag{2.2}$$
$$\mathbf{V}_1|_S = W_1\mathbf{n}, \quad \mathbf{V}_1|_\infty = 0$$

Varying (1.2) and using the boundary conditions and Gauss's theorem, we obtain

$$\delta D = -2\varepsilon \int_\Omega \mathbf{V}_1\cdot\Delta\mathbf{V}_0\, d\Omega - 4\varepsilon \int_S \frac{\partial V_{n0}}{\partial n} W_1\, dS \tag{2.3}$$

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